Energy-based Control of Bidirectional Vehicle Strings

Steffi Knorn, Alejandro Donaire, Juan C. Agüero and Richard H. Middleton

Abstract— **It is shown how some classes of symmetric bidirectional heterogeneous vehicle strings can be modelled using Hamiltonian functions. Hamiltonian systems theory is applied to show stability and string stability of the vehicle string.**

I. Introduction

In the field of coordinated systems, formation control is one of many control objectives. A group of N vehicles (e. g. platoon or string) is required to follow a given reference trajectory while the vehicles keep a prescribed distance to neighbouring vehicles. In its simplest form the vehicles in the platoon are only considered to move in one direction.

Even though it is possible to create a decentralised controller for the entire string, see e. g. [1], it is usually desirable to find distributed control solutions where each vehicle is equipped with a local controller using local measurements. In case information is only propagating through the string in one direction the string is called unidirectional, e. g. [2], [3], and bidirectional otherwise, e. g. [4], [5]. A string is called homogeneous if the dynamics of each vehicle and its controller are independent of its position within the string, e. g. [4], and heterogeneous otherwise, e. g. [6].

In most cases it is straight forward to design local controller to achieve a stable string in the usual sense. Thus small initial deviations or disturbances cause small perturbations. However, it is well known that error signals can amplify when travelling through the string resulting in growth of the local error norm with the position in the string. This effect is referred to as string instability', e.g. in [4], [7], or 'slinky effect', e.g. in [8].

It was shown in [4], [9] that similar to unidirectional strings, linear, symmetric bidirectional strings with two integrators in the open loop and constant spacing are always string unstable. [7] examines a bidirectional string with constant spacing and shows that string stability can be achieved with sufficiently large coupling with the leader position. The authors of [5] approximate a linear, bidirectional string of N vehicles as a PDE. It is shown that the least stable eigenvalue of the PDE approaches the origin with $O(1/N^2)$ if the string is symmetric and $O(1/N)$ if the string is asymmetric. However, the knowledge of the reference velocity is needed.

A different approach was considered in [10]. Modelling a symmetric bidirectional string as a mass-spring-damper system, it is shown that string stability with constant spacing can be guaranteed if the damping coefficients or the inverse compliances of the dampers or springs between the vehicles, respectively, grow with the string length N. When using a velocity depending distance between the vehicles, string stability can be guaranteed when the time headway is larger than a infimal time headway (independently of the string length N). In [8] the authors consider a bidirectional heterogeneous string of vehicles using a detailed nonlinear vehicle including aerodynamic drag and friction forces. Using the relative velocity error and position error towards the predecessor and follower string stability can be guaranteed using a time headway policy. Sufficient conditions for string instability of bidirectional, heterogeneous strings were derived in [11]. These results are used to derive an infimal average time headway that may avoid string instability.

A range of different methods has been used in the literature so far to analyse vehicle platoons. The Laplace transform with respect to time is used for instance in [4], [10], [11] to analyse the system dynamics in the frequency domain. Lyapunov Theory has been applied in [8] and graph theory was used in [6] to analyse a string of vehicles with a general interconnection or communication structure. In [5] a bidirectional string was approximated as a PDE.

In this paper we will propose to use port Hamiltonian systems (PHS), to study a bidirectional string modelled as a mass-spring-damper system. One clear advantage of this method is that the model is based on physically meaningful states and therefore yields a direct physical interpretation of the energy of the system. As the Hamiltonian function can be seen as the energy storage function conditions for standard stability and L_2 string stability of the system can be derived directly from the Hamiltonian function without lengthy or cumbersome analysis as for instance in [5], [10]. Another major advantage of using port-Hamiltonian systems is that studying heterogeneous strings does not complicate the analysis (compared to the analysis using a PDE approximation in [5] or transfer functions as in [4], [10]). Also, it is easy and straight forward to extend the stability analysis proposed in this work to nonlinear strings. add more stuff on PHS

Local control using virtual springs and dampers between the vehicles will be considered in combination with a drag force towards the ground. Integral action will be introduced to enhance the performance of the system. add more stuff on IA

As it was noted above it has been shown that string stability cannot be achieved for linear symmetric bidirectional strings with two integrators in each vehicle, see [4], [9], we will relax the definition for string stability: Instead of requiring the L_2 norm of all states (e. g. $\|x(\cdot)\|_2 = \sqrt{\int_0^\infty |x(t)|^2 \mathrm{d}t}$)

S. Knorn, A. Donaire, J.C. Agüero and R.H. Middleton are with the Centre for Complex Dynamic Systems and Control, The University of Newcastle, NSW 2308, Australia; Email: steffi.knorn@newcastle.edu.au, alejandro.donaire@newcastle.edu.au, juan.aguero@newcastle.edu.au and richard.middleton@newcastle.edu.au

to be bounded for all N, we will focus on the point wise norm $|x(t)| = \sqrt{x^{\mathrm{T}}(t)x(t)}$ and require $|x(t)| < \infty$ for string stability. This is a reasonable choice as it guarantees that the local separation errors between the vehicles are bounded at all times. Thus, it can be guaranteed that the cars do not crash into each other.

After discussing the notation and problem description in Section II, local control between the vehicles will be introduced in Section III. As local control does not guarantee string stability, integral action control will be introduced in Section III. A short discussion of nonlinear strings in Section V is followed by an illustrative example in Section VI and concluding remarks.

II. Problem Formulation

A. Notation

We consider a system of N vehicles with mass m_i. The motion equations of the system can be described using the momentum and position of each vehicle, i.e. p_i and q_i with $i = 1,2,\cdots,N$, as follows

$$\dot{p}_i = F_i + d_i \tag{1}$$

$$\dot{q}_i = m_i^{-1} p_i \tag{2}$$

where F_i is the control force on the vehicle and d_i is the disturbance. The control force F_i will be chosen such that only data of a group of nearest neighbours of the ith vehicle (both preceding and following vehicles) are needed. When denoting all local states of the ith vehicle (i. e. its momentum, its position and possible controller states) by x_i the dynamics of the ith vehicle within the string are given by

$$\dot{x}_i = \begin{cases} f_i(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+k_r}, d_i) & \text{if } i \leq k_{\mathrm{f}} \\ f_i(x_{i-k_f}, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+k_r}, d_i) & \text{if } k_{\mathrm{f}} < i < N - k_{\mathrm{r}} \\ f_i(x_{i-k_f}, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N, d_i) & \text{if } i \geq N - k_{\mathrm{r}} \end{cases} \tag{3}$$

for $i = 1,\ldots,N$ with initial conditions $x_{i_0} = x_i(t = 0)$ where $k_{\mathrm{f}} \in \mathbb{N}$ is the forward communication range, $k_{\mathrm{r}} \in \{\mathbb{N}, 0\}$ is the rear communication range and x_0 is a reference signal. We assume $N > k_{\mathrm{f}} + k_{\mathrm{r}}$ and that $f_i(0,\ldots,0) = 0$ for all i.

We denote the state and the disturbance vector by the column vectors $x(t) = \mathrm{col}(x_1(t),\ldots,x_N(t))$ and $d(t) = \mathrm{col}(d_1(t),\ldots,d_N(t))$, i. e.

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{pmatrix}, \quad x_0 = \begin{pmatrix} x_{1_0} \\ \vdots \\ x_{N_0} \end{pmatrix} \quad \text{and} \quad d(t) = \begin{pmatrix} d_1(t) \\ \vdots \\ d_N(t) \end{pmatrix}. \tag{4}$$

The column vector of ones is denoted by $\underline{1}$ and $\vec{e}_i$ is the ith canonical vector of length N. Similarly we denote the diagonal matrix $A \in \mathbb{R}^{N\times N}$ with diagonal entries $a_1,\ldots a_N$ as $A = \mathrm{diag}(a_1,\ldots a_N)$.

B. Control Objectives

The local control objective for each vehicle is to bring its local error to zero using local (distributed) control and only locally available data. In a unidirectional string the local error is the position error towards one or more preceding vehicles. Here we consider a bidirectional string where the local controller is driven by both the position error towards a group of preceding and following vehicles. The controller of the first vehicle in the string aims to follow a given trajectory q_0 and also minimise the local position error towards a group of following vehicles. In the simplest setting the reference signal is considered to be a ramp with constant velocity v_0, i.e. $q_0 = v_0 t$.

Note that the vehicles within the string (apart from a limited group at the beginning of the string) do not have access to the reference signal and therefore have to adjust their position and momentum indirectly by forcing their local position error to zero.

The overall control objective is to achieve “string stability” or “scalability”. This is that the norm of the local states of the complete string do not grow without bound as N increases for nonzero disturbances or initial conditions. We will make use of the following definition for string stability:

Definition 1 (String Stability): The system (3) with equilibrium x^* is string stable if there exists a $k < \infty$ such that for all N and all $i = 1,\ldots,N$ $|x_i^*| \leq k$ and for all $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$\|d(\cdot)\|_2 < \delta(\epsilon) \quad \text{and} \quad |x_0 - x^*| < \delta(\epsilon) \tag{5}$$

implies

$$|x(t) - x^*|_2 < \epsilon \quad \text{for all } t \geq 0 \text{ and } N \geq 1. \tag{6}$$

Note that the definition for string stability above requires two important properties of the system. First, the equilibrium states do not grow without bound as N increases. Furthermore, string stability guarantees that the states of the system remain bounded.

Note that instead of using the point wise (local) norm of $x(t) - x^*$ different definitions of string stability use the L_2 vector function norm $\|x(\cdot) - x^*\|$. However, it has been shown in [4] that every symmetric homogeneous bidirectional string with tight spacing and two poles in the open loop of each vehicle in the string is string unstable. Therefore, we will focus on the less restrictive definition stated above.

C. The Uncontrolled System

Since the control objectives are described in terms of the vehicle momenta, which is proportional to the velocities, and the distances between the vehicles, we will write the model using these variables as the states:

$$p_i = m_i v_i \tag{7}$$

and the local position error between the ith vehicle and its direct predecessor

$$\Delta_i = q_{i-1} - q_i \tag{8}$$

for $i = 1,\cdots,N$. The position q_0 is the product of the constant velocity reference v_0 and time. The state equation for the momenta is (1), and the dynamic equations for the local position error according to (2)

$$\dot{\Delta}_i = \dot{q}_{i-1} - \dot{q}_i = m_{i-1}^{-1} p_{i-1} - m_i^{-1} p_i \tag{9}$$

The dynamics of the string system described in momenta of the vehicles and separation distance between the vehicles can be described in Port-Hamiltonian form as

$$\begin{bmatrix}\dot{p}\\ \dot{\Delta}\end{bmatrix} = \begin{bmatrix}0 & S^{\mathrm{T}}\\ -S & 0\end{bmatrix}\nabla H(p,\Delta) + \begin{bmatrix}F\\ 0\end{bmatrix} + \begin{bmatrix}d\\ \vec{e}_1 v_0\end{bmatrix}, \tag{10}$$

where $\Delta, p \in \mathbb{R}^N$ are the displacement and momentum vector, i. e. $\Delta = \mathrm{col}(\Delta_1, \ldots, \Delta_N)$, $p = \mathrm{col}(p_1, \ldots, p_N)$, and the control force vector is $F = \mathrm{col}(F_1, \ldots, F_N)$.

The function H is the Hamiltonian function, and is given by

$$H(p,\Delta) = \frac{1}{2} p^{\mathrm{T}} M^{-1} p, \tag{11}$$

The matrix $M \in \mathbb{R}^{N\times N}$ is the constant and positive definite inertia matrix $M = \mathrm{diag}(m_1, \ldots, m_N)$. The matrix S has the bidiagonal form

$$S = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0\\ -1 & 1 & \ddots & & \vdots\\ 0 & -1 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1 & 0\\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}. \tag{12}$$

III. Local Control

Local (distributed) control between the vehicles will be introduced in this section. The control forces consist of the "spring force" F_i^{s}, that depends linearly on the position errors Δ_i, the "damper force" F_i^{r}, that depends linearly on the velocity errors between two neighbouring vehicles, and the "drag force" F_i^{d} describing the friction of vehicle i towards the ground:

$$\begin{aligned} F_i =& F_i^{\mathrm{s}} - F_{i+1}^{\mathrm{s}} - F_i^{\mathrm{d}} + F_i^{\mathrm{r}} - F_{i+1}^{\mathrm{r}}\\ =& c_i^{-1}\Delta_i - c_{i+1}^{-1}\Delta_{i+1} - b_i m_i^{-1} p_i + r_i(m_{i-1}^{-1} p_{i-1} - m_i^{-1} p_i)\\ & -r_{i+1}(m_i^{-1} p_i - m_{i+1}^{-1} p_{i+1}), \quad \forall \quad i = 1, \cdots, N-1,\\ F_N =& F_N^{\mathrm{s}} + F_N^{\mathrm{r}} - F_N^{\mathrm{d}}\\ =& c_N^{-1}\Delta_N + r_N(m_{N-1}^{-1} p_{N-1} - m_N^{-1} p_N) - b_N m_N^{-1} p_N \end{aligned} \tag{13}$$

such that we can write

$$F = -(B+R)M^{-1}p + \vec{e}_1 r_1 v_0 + S^{\mathrm{T}} C^{-1}\Delta \tag{14}$$

whith

$$R = \begin{bmatrix} r_1 + r_2 & -r_2 & 0 & \cdots & 0\\ -r_2 & r_2 + r_3 & -r_3 & \ddots & \vdots\\ 0 & -r_3 & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & r_{N-1} + r_N & -r_N\\ 0 & \cdots & 0 & -r_N & r_N \end{bmatrix}, \tag{15}$$

$B = \mathrm{diag}(b_1, \ldots, b_N)$ and $C = \mathrm{diag}(c_1, \ldots, c_N)$.

We will show that the system is asymptotically stable with respect to the equilibrium (p^*, Δ^*). However, the values of the displacements in steady state are undesirable and grow with the string length N in presence of a nonzero reference velocity v_0.

Lemma 1: Consider the string system (10) in closed loop with the control law (14). Then,

(i) the equilibrium $(p^*, \Delta^*) = (M\underline{1}v_0, CS^{-\mathrm{T}}B\underline{1}v_0)$ is global asymptotically stable in the absence of disturbances, i.e. $d = 0$; and

(ii) the system is passive with input d, output $y = \nabla_p H_{\mathrm{cl}}(p,r)$ and storage function H_{cl}.

Proof: (*i*): From (14) and (10) the dynamic equations for the closed loop have the form

$$\begin{aligned} \dot{p} =& S^{\mathrm{T}} C^{-1}\Delta - RM^{-1}p + \vec{e}_1 r_1 v_0 - BM^{-1}p + B\underline{1}v_0 - B\underline{1}v_0 + d\\ =& -(R+B)M^{-1}(p - M\underline{1}v_0) + S^{\mathrm{T}}C^{-1}(\Delta - CS^{-\mathrm{T}}B\underline{1}v_0) + d, \end{aligned} \tag{16}$$

$$\begin{aligned} \dot{\Delta} =& -SM^{-1}p + \vec{e}_1 v_0\\ =& -SM^{-1}(p - M\underline{1}v_0). \end{aligned} \tag{17}$$

Thus the closed loop has the port Hamiltonian form

$$\begin{bmatrix}\dot{p}\\ \dot{\Delta}\end{bmatrix} = \begin{bmatrix}-(B+R) & S^{\mathrm{T}}\\ -S & 0\end{bmatrix}\nabla H_{\mathrm{cl}}(p,\Delta) + \begin{bmatrix}d\\ 0\end{bmatrix}, \tag{18}$$

with the closed-loop Hamiltonian function

$$\begin{aligned} H_{\mathrm{cl}}(p,\Delta) =& \frac{1}{2}(p - M\underline{1}v_0)^{\mathrm{T}} M^{-1}(p - M\underline{1}v_0)\\ &+ \frac{1}{2}\left(\Delta - CS^{-\mathrm{T}}B\underline{1}v_0\right)^{\mathrm{T}} C^{-1}\left(\Delta - CS^{-\mathrm{T}}B\underline{1}v_0\right). \end{aligned} \tag{19}$$

Using $H_{\mathrm{cl}}(p,\Delta)$ as Lyapunov function, and computing the time derivative of $H_{\mathrm{cl}}(p,\Delta)$ along the solution of (18) setting $d = 0$ yields

$$\begin{aligned} \dot{H}_{\mathrm{cl}}(p,\Delta) =& \nabla^{\mathrm{T}} H_{\mathrm{cl}} \begin{bmatrix}-(B+R) & S^{\mathrm{T}}\\ -S & 0\end{bmatrix}\nabla H_{\mathrm{cl}}\\ =& -\nabla_p^{\mathrm{T}} H_{\mathrm{cl}}(B+R)\nabla_p H_{\mathrm{cl}}\\ \le& \, 0 \end{aligned} \tag{20}$$

since $(B+R) = (B+R)^{\mathrm{T}} > 0$. Since the biggest invariant set included in $\mathcal{S} = \{(p,\Delta) | \dot{H}_{\mathrm{cl}}(p,\Delta) = 0\}$ is $(p^*, \Delta^*) = (M\underline{1}v_0, CS^{-\mathrm{T}}B\underline{1}v_0)$. Thus, by LaSalle's Invariance Principle (found in most textbook, see e. g. [12, Theorem 4.4]) it can be shown that the system is asymptotically stable and the equilibrium reached is (p^*, Δ^*).

(*ii*): Using $H_{\mathrm{cl}}(p,\Delta)$ as Lyapunov function, and computing the time derivative of $H_{\mathrm{cl}}(p,\Delta)$ along the solution of (18) considering a nonzero disturbance yields

$$\begin{aligned} \dot{H}_{\mathrm{cl}}(p,\Delta) =& \nabla^{\mathrm{T}} H_{\mathrm{cl}}\left(\begin{bmatrix}-(B+R) & S^{\mathrm{T}}\\ -S & 0\end{bmatrix}\nabla H_{\mathrm{cl}} + \begin{bmatrix}d\\ 0\end{bmatrix}\right)\\ =& -\nabla_p^{\mathrm{T}} H_{\mathrm{cl}}(B+R)\nabla_p H_{\mathrm{cl}} + \nabla_p^{\mathrm{T}} H_{\mathrm{cl}} d. \end{aligned} \tag{21}$$

With $y = \nabla H_{\mathrm{cl}}$ this yields

$$\begin{aligned} \dot{H}_{\mathrm{cl}}(p,\Delta) =& -y^{\mathrm{T}}(B+R)y + y^{\mathrm{T}}d\\ \le& -\lambda_{\min}(B+R)|y|^2 + y^{\mathrm{T}}d\\ =& -\frac{\lambda_{\min}(B+R)}{2}|y|^2 + \frac{1}{2\lambda_{\min}(B+R)}|d|^2\\ & -\frac{\lambda_{\min}(B+R)}{2}\left|y - \frac{1}{\lambda_{\min}(B+R)}d\right|^2\\ \le& \frac{1}{2\lambda_{\min}(B+R)}|d|^2 \end{aligned} \tag{22}$$

which implies

$$\begin{aligned}H_{\mathrm{cl}}(p(t),\Delta(t)) \leq& H_{\mathrm{cl}}(p(0),\Delta(0)) + \frac{1}{2\lambda_{\min}(B+R)}\int_0^t |d(t)|^2 \mathrm{d}t\\ \leq& H_{\mathrm{cl}}(p(0),\Delta(0)) + \frac{1}{2\lambda_{\min}(B+R)}\|d(\cdot)\|^2. \quad (23)\end{aligned}$$

Thus the system is passive with input d, output $y = \nabla_p H_{\mathrm{cl}}(p,\Delta)$ and storage function H_{cl}. ∎

Note that the system is asymptotically stable with respect to the equilibrium $CS^{-\mathrm{T}}B\underline{1}v_0$ but not string stable according to Definition 1. From

$$S^{-\mathrm{T}} = \begin{bmatrix} 1 & 1 & \dots & 1\\ 0 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1\\ 0 & \dots & 0 & 1\end{bmatrix} \quad (24)$$

and $\Delta^* = CS^{-\mathrm{T}}B\underline{1}v_0$ we see that $\Delta_i^* = c_i\sum_{k=i}^N b_k v_0$. Thus, if a positive lower bound on the drag and compliance coefficients exist, i. e. $\min_i b_i > \underline{b} > 0$ and $\min_i c_i > \underline{c} > 0$, the steady state value of Δ_1 grows with N and therefore the system is not string stable.

This effect could be avoided by choosing parameters b_i and c_i that decrease sufficiently fast with i. However, choosing decreasing c_i implies that the parameters tend to zero at the end of strings of increasing length N. This implies that such a vehicle string is not scalable from a practical setting and is therefore undesirable. Choosing decreasing b_i also leads to string instability as the minimal eigenvalue of $B+R$ tends to zero as N increases. Thus, there does not exist an upper bound of the right hand side of (45).

IV. Integral Action

Before introducing integral action, we will prove the following Lemma that will be used later in this section.

Lemma 2: For any N and any choice of $B = \mathrm{diag}(b_1,\dots,b_N)$ and R as in (15) for $b_i,r_i > 0$ for all $i \in \{1,\dots,N\}$, there exits a $\delta < \infty$ such that

$$\left|\vec{e}_i^{\mathrm{T}}(B+R)^{-1}B\underline{1}\right| < \delta \quad (25)$$

for all $i \in \{1,\dots,N\}$.

Proof: Note that

$$(B+R)^{-1}B = I - (B+R)^{-1}R. \quad (26)$$

Thus,

$$\begin{aligned}\left|\vec{e}_i^{\mathrm{T}}(B+R)^{-1}B\underline{1}\right| \leq& \left|\vec{e}_i^{\mathrm{T}}\underline{1} - \vec{e}_i^{\mathrm{T}}(B+R)^{-1}R\underline{1}\right|\\ \leq& 1 + \left|\vec{e}_i^{\mathrm{T}}(B+R)^{-1}r_1\vec{e}_1\right|. \quad (27)\end{aligned}$$

Using Geršgorin's Theorem (see e. g. [13]) it is easy to show that all eigenvalues of R lie in the range $0 \leq \lambda_i(R) \leq$. Hence, R is a positive semidefinite matrix while B is a positive definite matrix. Therefore, using again Geršgorin's Theorem the minimal singular value of $B+R$ yields

$$\begin{aligned}\sigma_{\min}(B+R) =& \lambda_{\min}(B+R)\\ \geq& \min\begin{Bmatrix} b_1 + r_1 + r_2 - r_2;\\ \min_{1<i<N}\{b_i + r_i + r_{i-1} - r_i - r_{i-1}\};\\ b_n + r_N - r_N\end{Bmatrix}\\ \geq& \min_i b_i. \quad (28)\end{aligned}$$

Hence, $\sigma_{\max}\left((B+R)^{-1}\right) \leq (\min_i b_i)^{-1}$. Together with the fact that $\sigma_{\max}^2(A) \geq \frac{y^{\mathrm{T}}A^{\mathrm{T}}Ay}{y^{\mathrm{T}}y}$ holds for any vector $y \neq 0$, setting $A = (B+R)^{-1}$ and $y = r_1\vec{e}_1$ yields

$$\vec{e}_1^{\mathrm{T}}r_1(B+R)^{-\mathrm{T}}(B+R)^{-1}r_1\vec{e}_1 \leq r_1^2\sigma_{\max}^2\left((B+R)^{-1}\right) \leq r_1^2\min_i b_i^{-2}.$$

Finally note that

$$\left(\vec{e}_i - (B+R)^{-1}r_1\vec{e}_1\right)^{\mathrm{T}}\left(\vec{e}_i - (B+R)^{-1}r_1\vec{e}_1\right) \geq 0 \quad (29)$$

yields

$$\begin{aligned}\left|\vec{e}_i^{\mathrm{T}}(B+R)^{-1}r_1\vec{e}_1\right| \leq& \frac{\vec{e}_i^{\mathrm{T}}\vec{e}_i}{2} + \frac{\vec{e}_1^{\mathrm{T}}r_1(B+R)^{-\mathrm{T}}(B+R)^{-1}r_1\vec{e}_1}{2}\\ \leq& \frac{1}{2} + \frac{r_1^2}{2\,(\min_i b_i)^2} \quad (30)\end{aligned}$$

and therefore there exists an upper bound for each element of the vector, i. e. $|\vec{e}_i^{\mathrm{T}}(B+R)^{-1}B\underline{1}| < \delta = 1.5 + \frac{r_1^2}{2(\min_i b_i)^2}$ for all i. Note that the inequality is strict as the left side of (29) is only zero if $(B+R)^{-1}r_1\vec{e}_1 = \vec{e}_i$, which would lead to

$$\begin{aligned}\vec{e}_i^{\mathrm{T}}(B+R)^{-1}B\underline{1} =& \vec{e}_i^{\mathrm{T}}\underline{1} - \vec{e}_i^{\mathrm{T}}(B+R)^{-1}r_1\vec{e}_1\\ =& \vec{e}_i^{\mathrm{T}}\left(\underline{1} - \vec{e}_i\right)\\ =& 0. \quad (31)\end{aligned}$$

Thus, $\left|\vec{e}_i^{\mathrm{T}}(B+R)^{-1}B\underline{1}\right|$ is bounded. ∎

Lemma 3: Consider the string system (10) with a reference signal with constant velocity v_0, with constant disturbances d in closed loop with a controller obtained by adding the control in Lemma 1 and the dynamic controller

$$F_{\mathrm{IA}} = MKS^{\mathrm{T}}C^{-1}\Delta - (B+R)Kz_3 \quad (32)$$
$$\dot{z}_3 = -S^{\mathrm{T}}C^{-1}\Delta. \quad (33)$$

where $K \in \mathbb{R}^{N\times N}$ is a diagonal positive matrix $K = \mathrm{diag}(k_1,\dots,k_N)$. Then

(i) the desired equilibrium point

$$(p^*,\Delta^*,z_3^*) = (M\underline{1}v_0,0,\alpha) \quad (34)$$

with $\alpha = K^{-1}(B+R)^{-1}(d - B\underline{1}v_0)$ is globally asymptotically stable (despite the presence of constant unknown disturbances),

(ii) the system is passive with input d, output $y = \nabla_{z_1}H_z(z)$ and storage function H_z, and

(iii) the system is string stable.

Proof: (i): We will use the following change of coordinates

$$z_1 = p - M\underline{1}v_0 + MK(z_3 - \alpha), \quad (35)$$
$$z_2 = \Delta \quad (36)$$

Thus, combining (16), (32) and (33) yields

$$\begin{aligned}\dot{z}_1 =& \dot{p} + MK\dot{z}_3\\ =& S^{\mathrm{T}}C^{-1}\Delta - RM^{-1}p + \vec{e}_1 r_1 v_0 - BM^{-1}p + d\\ &+MKS^{\mathrm{T}}C^{-1}\Delta - (B+R)Kz_3 - MKS^{\mathrm{T}}C^{-1}\Delta\\ =& S^{\mathrm{T}}C^{-1}\Delta - RM^{-1}p + \vec{e}_1 r_1 v_0 - BM^{-1}p + d\\ &-(B+R)Kz_3 + B\underline{1}v_0 - B\underline{1}v_0\\ =& -(B+R)M^{-1}(p - M\underline{1}v_0 + MK(z_3-\alpha)) + S^{\mathrm{T}}C^{-1}\Delta\\ =& -(B+R)M^{-1}z_1 + S^{\mathrm{T}}C^{-1}z_2 \end{aligned} \tag{37}$$

and (17) yields

$$\begin{aligned}\dot{z}_2 =& \dot{\Delta}\\ =& -SM^{-1}(p - M\underline{1}v_0)\\ =& -SM^{-1}(p - M\underline{1}v_0 + MK(z_3-\alpha)) + SK(z_3-\alpha)\\ =& -SM^{-1}z_1 + SK(z_3-\alpha).\end{aligned} \tag{38}$$

Thus, the closed loop dynamics have the port Hamiltonian form

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix} = \begin{bmatrix}-(B+R) & S^{\mathrm{T}} & 0\\ -S & 0 & S\\ 0 & -S^{\mathrm{T}} & 0\end{bmatrix}\nabla H_z(z) \tag{39}$$

with the Hamiltonian function

$$H_z(z) = \frac{1}{2}z_1^{\mathrm{T}}M^{-1}z_1 + \frac{1}{2}z_2^{\mathrm{T}}C^{-1}z_2 + \frac{1}{2}(z_3-\alpha)^{\mathrm{T}}K(z_3-\alpha). \tag{40}$$

Using $H_z(z)$ as Lyapunov function, and computing the time derivative of $H_z(z)$ yields

$$\begin{aligned}\dot{H}_z(z) =& \nabla^{\mathrm{T}}H_{\mathrm{cl}}\begin{bmatrix}-(B+R) & S^{\mathrm{T}} & 0\\ -S & 0 & S\\ 0 & -S^{\mathrm{T}} & 0\end{bmatrix}\nabla H_z(z)\\ =& -\nabla_{z_1}^{\mathrm{T}}H_z(B+R)\nabla_{z_1}H_z\\ \le& 0\end{aligned} \tag{41}$$

since $(B+R) = (B+R)^{\mathrm{T}} > 0$. Since the biggest invariant set included in $\mathcal{S} = \{z|\dot{H}_z(z) = 0\}$ is $(z_1^*,z_2^*,z_3^*) = (0,0,\alpha)$. Thus, by LaSalle's Invariance Principle (found in most textbook, see e. g. [12, Theorem 4.4]) it can be shown that the system is asymptotically stable and the equilibrium reached is (z_1^*,z_2^*,z_3^*). This implies that the equilibrium in the original coordinates is $(p^*,\Delta^*,z_3^*) = (M\underline{1}v_0,0,\alpha)$.

(ii): When choosing the Hamiltonian function (40) but setting

$$\beta = -K^{-1}(B+R)^{-1}B\underline{1}v_0, \tag{42}$$

the closed loop dynamics have the port Hamiltonian form

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix} = \begin{bmatrix}-(B+R) & S^{\mathrm{T}} & 0\\ -S & 0 & S\\ 0 & -S^{\mathrm{T}} & 0\end{bmatrix}\nabla H_{z2}(z) + \begin{pmatrix}d\\0\\0\end{pmatrix}. \tag{43}$$

with

$$H_{z2}(z_1,z_2,z_3) = \frac{1}{2}z_1^{\mathrm{T}}M^{-1}z_1 + \frac{1}{2}z_2^{\mathrm{T}}C^{-1}z_2 + \frac{1}{2}(z_3-\beta)^{\mathrm{T}}K(z_3-\beta).$$

Hence, following similar steps as in (22) this yields

$$\dot{H}_{z2}(z_1,z_2,z_3) \le \frac{1}{2\lambda_{\min}(B+R)}|d|^2 \tag{44}$$

which implies

$$H_{z2}(z(t)) \le H_{z2}(z(0)) + \frac{1}{2\lambda_{\min}(B+R)}\|d(\cdot)\|^2. \tag{45}$$

Thus, the system is passive with input d, output $y = \nabla_{z_1}H_z(z)$ and storage function H_z.

(iii): Note that the equilibrium states p^* and Δ^* given in (34) are bounded since the matrices M, C and B are diagonal with positive bounded entries for any string length N. Lemma 2 together with the fact that K is diagonal with positive bounded entries ensures that $z_3^* = \alpha$ is also bounded element wise independently of N.

With (28) equation (45) can be bounded by

$$\begin{aligned}H_{z2}(z(t)) \le& (\min_i(m_i))^{-1}|z_1(0)|^2 + (\min_i(c_i))^{-1}|z_2(0)|^2\\ &+ \max_i k_i|z_3-\alpha|^2 + (2\min_i(b_i))^{-1}\|d(\cdot)\|^2.\end{aligned} \tag{46}$$

Since the mass m_i, the compliance c_i, the drag coefficient b_i and the integral action control parameter k_i for each vehicle are positive the norm of the states z is bounded for all N if $|z(0)|$ and $\|d(\cdot)\|_2$ do not increase with N. Therefore, the system is string stable according to Definition 1. ■

V. Nonlinear Control

One important advantage of modelling the bidirectional string as a port-Hamiltonian system is that it is extendable to nonlinear systems, i. e. strings with nonlinear control forces due to nonlinear springs and dampers. As a proof of concept we will shortly discuss a simple local control law with arbitrary nonlinear spring forces (satisfying the condition $f_i^{\mathrm{s}}(\Delta_i)\Delta_i \ge 0$ and $f_i^{\mathrm{s}}(\Delta_i) = 0$ only for $\Delta_i = 0$).

Nonlinear springs yield some significant advantages over linear spring models. For instance ... list some advantages here

Assume the spring force between vehicle $i-1$ and i is given by the nonlinear function $f_i^{\mathrm{s}}(\Delta_i)$ with $f_i^{\mathrm{s}}(\Delta_i)\Delta_i \ge 0$ and $f_i^{\mathrm{s}}(\Delta_i) = 0$ only for $\Delta_i = 0$ for all $i = 1,\ldots,N$. Thus,

$$F^{\mathrm{s}} = S^{\mathrm{T}}f^{\mathrm{s}}(\Delta) \tag{47}$$

where $f^{\mathrm{s}}(\Delta)$ is the column vector with entries $f_1^{\mathrm{s}}(\Delta_1),\ldots,f_N^{\mathrm{s}}(\Delta_N)$.

Hence, the dynamics of a system with nonlinear spring forces and linear dampers (without drag forces), i. e. $F = S^{\mathrm{T}}f^{\mathrm{s}}(\Delta) - RM^{-1}p + R\underline{1}v_0$, can be described by

$$\begin{bmatrix}\dot{p}\\ \dot{\Delta}\end{bmatrix} = \begin{bmatrix}-R & S^{\mathrm{T}}\\ -S & 0\end{bmatrix}\nabla H_{\mathrm{cl}}(p,\Delta) + \begin{bmatrix}d\\0\end{bmatrix} \tag{48}$$

with

$$H_{\mathrm{cl}}(p,\Delta) = \frac{1}{2}(p - M\underline{1}v_0)^{\mathrm{T}}M^{-1}(p - M\underline{1}v_0) + \sum_{i=1}^{N}\int_0^{\Delta_i} f_i^{\mathrm{s}}(w)\mathrm{d}w. \tag{49}$$

Therefore the equilibrium $(p^*,\Delta^*) = (M\underline{1}v_0,0)$ is asymptotically stable.

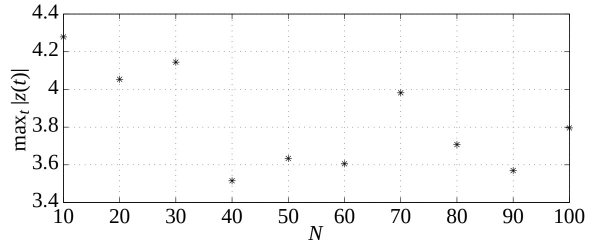

Fig. 1: Homogeneous string: $\max_t |z(t)|$

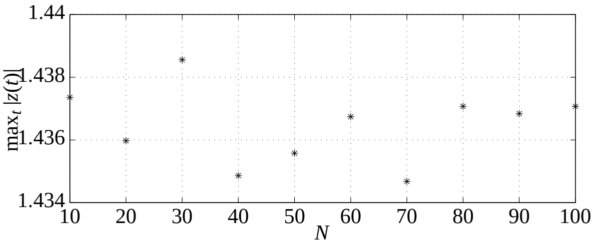

Fig. 2: Heterogeneous string: $\max_t |z(t)|$

VI. Example

Ten homogeneous strings of length $N = 10, 20, \ldots, 100$ with local control and integral action control have been simulated. Random nonzero initial conditions for p and δ normalised such that $|p(0) - M\underline{1}v_0| + |\Delta(0)| = 1$ haven been used. Also, an exponentially decaying disturbance onto the first vehicle in the string has been chosen such that $\|d(\cdot)\|_2 = 1$.

First a homogeneous string with the following parameters has been simulated: $m_i = 1$, $c_i = 1$, $b_i = 0.1$, $r_1 = 20$ and $K_i = 100$ for all $i = 1, 2, \ldots, N$. The maximum point wise norm of $z(t)$ for $N = 10, 20, \ldots, 100$ is shown in Figure 1.

In the second simulation a heterogeneous string with randomised parameters in the following range has been chosen: $m_i \in [1, 2]$, $c_i \in [1, 2]$, $b_i \in [0.1, 0.2]$, $r_1 \in [20, 21]$. The maximum point wise norm of $z(t)$ for $N = 10, 20, \ldots, 100$ is shown in Figure 2.

VII. Conclusions

It is shown how heterogeneous bidirectional vehicle strings can be described using physically meaningful Hamiltonian functions. This enables a simple and straight forward proof of asymptotical stability and string stability (scalability with respect to the string length N) of the resulting system.

As it has been shown in [4] that the commonly strict form of string stability (requiring the L_2 of all states to be bounded for any L_2 bounded disturbance) cannot be achieved for symmetric homogeneous bidirectional strings with tight spacing and two poles in the open loop of each vehicle in the string, the definition of string stability used here only requires the point wise (in time) norm of the states to be bounded for L_2 bounded disturbances. However, it might be possible to show string stability for some classes of bidirectional strings using the framework introduced here in the future.

Other possible future extensions of the results presented here include the discussion of a more general class of bidirectional vehicle strings, nonlinear strings (e.g. with nonlinear springs and dampers) and discrete time systems.

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